Holomorphic function spaces and the geometry of image domains

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This research has been funded by the PID2022-136320NB-I00 project: AEI/10.13039/501100011033, FEDER, UE

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Today: $X =$ Bloch $\mathcal{B}$, BMOA, Nevanlinna N, Smirnov N^+, Hardy H^p, Bergman A^p_α.

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$$\int_{\mathbb{D}} \left|f'(z)\right|^2 dA(z) < +\infty.$$

For every $z \in \mathbb{C}$ and every $R > 0$ there exists a map $f\colon \mathbb{D} \to D(z, R)$ such that $f \notin \mathcal{D}$.

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Consequence: There are no $\mathcal{D}$-domains.

Reformulation

Definition

We say that X enjoys the subordination property if $f \circ \varphi \in X$ whenever $f \in X$ and $\varphi\colon \mathbb{D} \to \mathbb{D}$ is holomorphic.

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Lemma

Assume that X enjoys the subordination property and that Ω is hyperbolic (i.e., its complement possesses at least two points). Then, the following are equivalent:

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- Some (hence, all) universal covering map $f_\Omega\colon \mathbb{D} \to \Omega$ belongs to X.

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Recall that, for a hyperbolic domain Ω, a universal covering map $f_\Omega \colon \mathbb{D} \to \Omega$ is a holomorphic map with the following property:
For every $f \in \mathrm{Hol}(\mathbb{D}, \Omega)$ there exists $\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ such that $f = f_\Omega \circ \varphi$.

$$\begin{tikzcd} & \mathbb{D} \arrow[d, "f_\Omega"] \\ \mathbb{D} \arrow[ur, "\varphi"] \arrow[r, "f"'] & \Omega \end{tikzcd}$$

PART I

Classical answers for classical spaces

Bloch

Let $\mathcal{B}$ be the Bloch space. That is, the space of all holomorphic maps on $\mathbb{D}$ for which

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Theorem (Seidel, Walsh – 1942)

Let Ω be a planar domain. For $z \in \Omega$, consider $R(z) = \sup\{r > 0 : D(z,r) \subset \Omega\}$. The following are equivalent:

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Examples. $\mathbb{C} \setminus (\mathbb{Z} \times \mathbb{Z})$ is a $\mathcal{B}$-domain ($R(z) \leq \sqrt{2}/2$).
Half-planes are not $\mathcal{B}$-domains.

BMOA

Let BMOA be the space of all holomorphic maps on $\mathbb{D}$ with bounded mean oscillation. Equivalently, athose for which

$$\sup_{w\in\mathbb{D}}\left(\sup_{r\in[0,1)}\int_0^{2\pi}|f_w(re^{i\theta})|^2 d\theta\right) < +\infty, \quad \text{where} \quad f_w(z) = f\left(\frac{z+w}{1+\overline{w}z}\right) - f(w), \quad z \in \mathbb{D}.$$

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Theorem (Hayman, Pommerenke – 1978)

Let Ω be a planar domain. The following are equivalent:

- Ω is a BMOA-domain.
- There exist $R, C>0$ such that, for all $z\in\Omega$, $\mathrm{cap}((\mathbb{C}\setminus\Omega)\cap D(z,R))\geq C$.

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Recall that, given a set $A\subset\mathbb{C}$, its logarithmic capacity is defined as

$$\mathrm{cap}(A)=\sup_{\mu}\left\{\exp\left(\iint\log|z-w|\,d\mu(z)d\mu(w)\right)\right\},$$

where μ is any probability whose support is a compact set lying on A.

A relation between Bloch and BMOA

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Corollary

Assume that Ω is simply connected. The following are equivalent:

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- Ω is a $\mathcal{B}$-domain.

Nevanlinna's class

Nevanlinna's class N is the space of all holomorphic maps on $\mathbb{D}$ with

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The following result for hyperbolic domains appears in Nevanlinna's book (1936):

- If $\operatorname{cap}(\mathbb{C} \setminus \Omega) = 0$, then f_Ω possesses radial limits almost nowhere (in particular, $f_\Omega \notin \mathrm{N}$).
- If $\operatorname{cap}(\mathbb{C} \setminus \Omega) > 0$, then $f_\Omega \in \mathrm{N}$ (in particular, it possesses radial limits almost everywhere).

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Corollary (Nevanlinna; Frostman)

Let Ω be a planar domain. The following are equivalent:

- Ω is a N-domain.
- The complement of Ω is not a polar set (i.e., its logarithmic capacity is positive).

Smirnov's class

Smirnov's class N^+ is the subspace of N for which

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For a planar domain Ω, a point $\xi \in \mathbb{C}_\infty \setminus \Omega$ is said to be regular for Ω if:

- Either $\xi \notin \partial\Omega$,
- or $\xi \in \partial\Omega$ and it is a regular point for Ω w.r.t. the Dirichlet problem (see Wiener's Criterion).

PART II

Recent contributions

The Hardy number

For $p > 0$, let the Hardy space H^p be the family of all holomorphic maps on $\mathbb{D}$ such that

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Properties. (1) If $\Omega_1 \subset \Omega_2$, then $\mathrm{h}(\Omega_1) \geq \mathrm{h}(\Omega_2)$. (2) $\mathrm{h}(a\Omega + b) = \mathrm{h}(\Omega)$ for $a, b \in \mathbb{C}$, $a \neq 0$.

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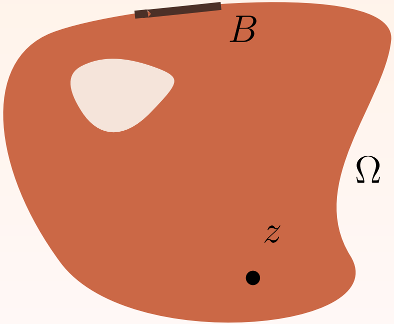
Note. Karafyllia (2020) constructed a H^p-domain with Hardy number p ($p = 3\pi$).

Definition

Let Ω be a planar domain whose complement is non-polar. The harmonic measure $\omega(z, B, \Omega)$ of $B \subset \partial\Omega$ at a point $z \in \Omega$ is the value at z of the solution of the generalized Dirichlet problem in Ω with boundary values 1 on B and 0 on $\partial\Omega \setminus B$.



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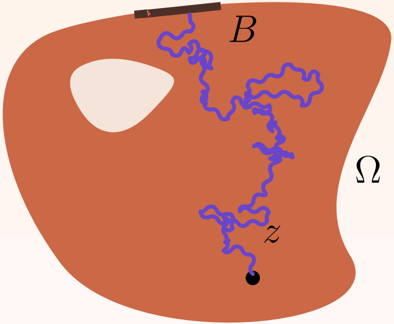

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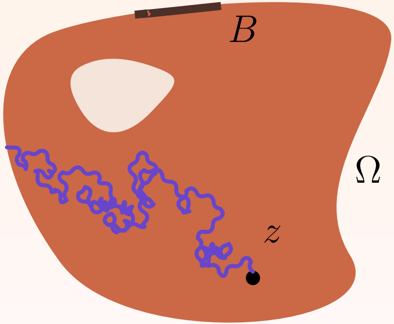

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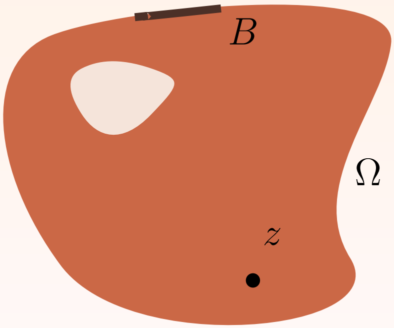
The map $\Omega \ni z \mapsto \omega(z, B, \Omega)$ is harmonic on Ω for every $B \subset \partial\Omega$.

The map $\partial\Omega \supset B \mapsto \omega(z, B, \Omega)$ is a probability measure for every $z \in \Omega$.

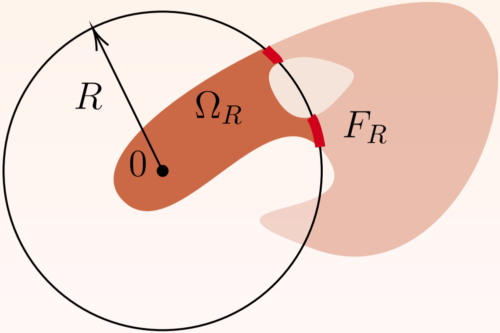


A description of the Hardy number of a domain

Theorem (Essén – 1981; Kim, Sugawa – 2011)

For a planar domain Ω with $0 \in \Omega$,

$$\mathrm{h}(\Omega) = \liminf_{R \to +\infty} \left(-\frac{\log \omega(0, F_R, \Omega_R)}{\log R} \right).$$



Ω_R is the connected component of $\Omega \cap \{z \in \mathbb{C} : |z| < R\}$ containing 0.

$F_R = \partial\Omega_R \cap \{z \in \mathbb{C} : |z| = R\}$

Simply connected domains

Lemma

For a hyperbolic domain Ω, $\mathrm{h}(\Omega) = \sup(\{0\} \cup \{p > 0 : f_\Omega \in \mathrm{H}^p\})$.

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Let $\theta \in (0, 2\pi]$. If $\Omega = \{z = re^{it} : r > 0, 0 < t < \theta\}$, $\theta \in (0, 2\pi]$, $\mathrm{h}(\Omega) = \pi/\theta$.

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Question

For every $p \in \{0\} \cup [1/2, +\infty]$ we know a planar domain whose Hardy number is p. What about $p \in (0, 1/2)$?

Filling the gap in $(0, 1/2)$

Theorem (Contreras, C-Z, Kourou, Rodríguez-Piazza – 2024)

For every $p \in (0, 1/2)$ there exists a domain $\Omega \subset \mathbb{C}$ such that $\mathrm{h}(\Omega) = p$.

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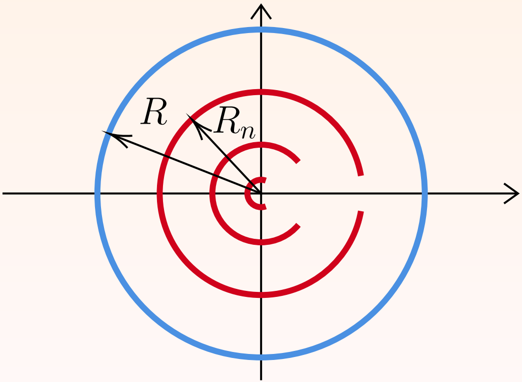
Construction.



Green function

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Let Ω be a planar domain. A Green function for Ω is a map $g_\Omega\colon \Omega \times \Omega \to (-\infty, +\infty]$ such that, for all $w \in \Omega$, the following properties hold:

1. $z \mapsto g_\Omega(z, w)$ is harmonic on $\Omega \setminus \{w\}$ and bounded outside every neighbourhood of w,
2. $g_\Omega(w, w) = +\infty$. Moreover, $g_\Omega(z, w) = -\log|z - w| + \mathrm{O}(1)$ as $z \to w$,
3. for nearly every $\xi \in \partial\Omega$, $g_\Omega(z, w) \to 0$ as $z \to \xi$.

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Example.

$$g_{\mathbb{D}}(z, w) = \log\left|\frac{1 - z\overline{w}}{z - w}\right|, \qquad z, w \in \mathbb{D}.$$

Theorem (Betsakos, C-Z – 2024)

Let Ω be a planar domain whose complement is non-polar. Assume that $0 \in \Omega$. Consider

$$\Psi_\Omega(R) = \int_{-\pi}^{\pi} g_\Omega(0, Re^{i\theta})d\theta, \qquad R > 0,$$

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Hardy number and Green function

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$$\mathrm{h}(\Omega) = \liminf_{R\to+\infty} \left(-\frac{\log(\Psi_\Omega(R))}{\log R} \right).$$

Hardy number and hyperbolic distance

Theorem (Karafyllia – 2020)

Let $\Omega \neq \mathbb{C}$ be a simply connected planar domain. Assume that $0 \in \Omega$. Then,

$$\mathrm{h}(\Omega) = \liminf_{R \to +\infty} \frac{d_\Omega(0, F_R)}{\log(R)},$$

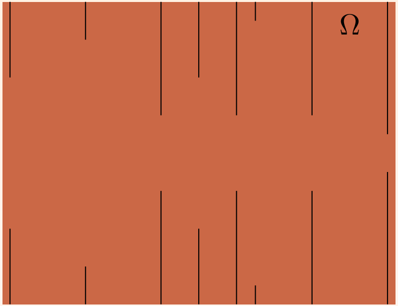
where $F_R = \{z \in \Omega : |z| = R\}$.

Domains with special geometric attributes - I

Let $(x_n)_{n\in\mathbb{Z}}$ be an increasing sequence of real numbers with no accumulation points, and let $(y_n)_{n\in\mathbb{Z}}$ be a sequence of positive numbers. Then, the planar domain

$$\Omega = \mathbb{C} \setminus \left(\bigcup_{n\in\mathbb{Z}} \{x_n + iy : |y| \geq y_n\} \right)$$

is called a comb domain.



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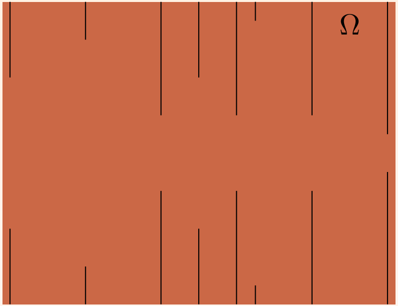

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If Ω is a comb domain, then $\mathrm{h}(\Omega) \geq 1$. Moreover, for every $p \in [1, +\infty]$ there exists a comb domain Ω with $\mathrm{h}(\Omega) = p$.

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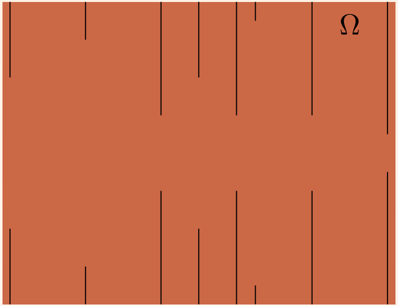

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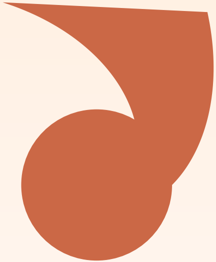
Connection: Exit time of Brownian motion.



A planar domain Ω is said to be spiral-like of order $\lambda = e^{i\phi}$, $-\pi/2 < \phi < \pi/2$, if $ze^{\lambda t} \in \Omega$ whenever $t \leq 0$ and $z \in \Omega$.

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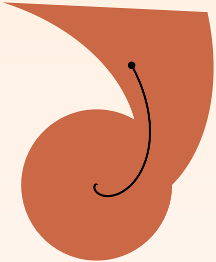

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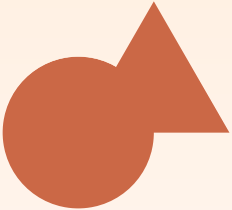

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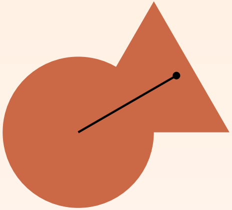

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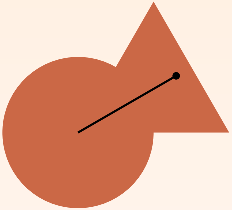

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Moreover, if $A > 0$, then Ω is not a $\mathrm{H}^{\mathrm{h}(\Omega)}$-domain.

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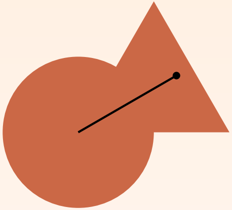

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Connection: Koenigs maps for elliptic dynamics in $\mathbb{D}$ (Poggi-Corradini).

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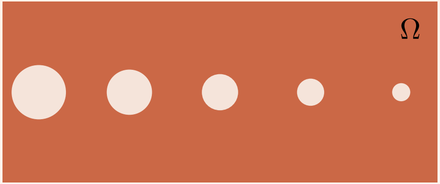

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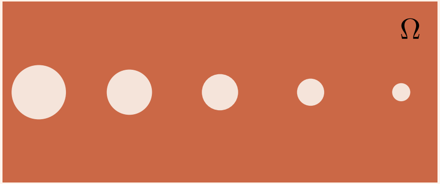

Theorem (Contreras, C-Z, Kourou, Rodríguez-Piazza – 2024)

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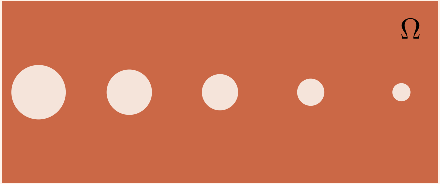

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Recall that

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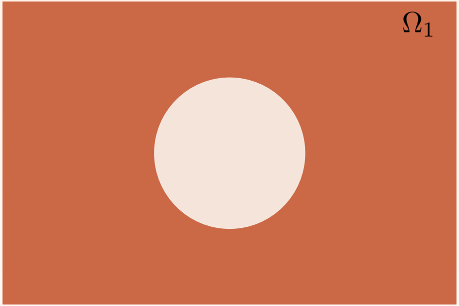
However, such inclusions are strict (even using universal covering maps!).



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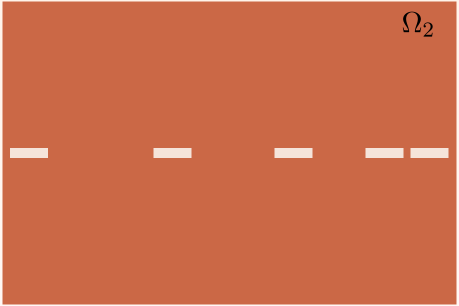

$f_{\Omega_2} \in \mathrm{N}^+$, $f_{\Omega_2} \notin \mathrm{H}^p$

For $p > 0$, let A^p be the Bergman space. That is, the collection of all holomorphic maps on $\mathbb{D}$ with

$$\int_{\mathbb{D}} |f(z)|^p \, dA(z) < +\infty.$$

For a planar domain Ω, it is possible to consider its 0-Bergman number as

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For $p > 0$ and $\alpha > -1$, let A^p_α be the Bergman space. That is, the collection of all holomorphic maps on $\mathbb{D}$ with

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Karafyllia (2023), after a collaboration with Karamanlis, introduced the Bergman number of Ω as

$$\mathrm{b}(\Omega) := \inf(\{\mathrm{b}(f) : f \in \mathrm{Hol}(\mathbb{D}, \Omega)\}),$$

where

$$\mathrm{b}(f) = \sup\left(\{0\} \cup \left\{\frac{p}{\alpha + 2} : \alpha > -1, p > 0, f \in \mathrm{A}^p_\alpha\right\}\right) \in [0, +\infty].$$

Relations among such numbers

Recall that $H^p \subset A^q_\alpha$ for $p \geq q/(\alpha+2)$. This means that

$$\mathrm{h}(\Omega) \leq \frac{\mathrm{b}_\alpha(\Omega)}{\alpha+2} \leq \mathrm{b}(\Omega).$$

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Remark: Such equalities may not hold in the general case: $\Omega = \mathbb{C} \setminus (\mathbb{Z} \times \mathbb{Z})$.

Strict inequalities

Theorem (Betsakos, C-Z – 2025)

- For every $p \in [0, +\infty)$ there exists a planar domain Ω with $p = \mathrm{h}(\Omega) < \mathrm{b}(\Omega) = \mathrm{b}_\alpha(\Omega) = +\infty$.

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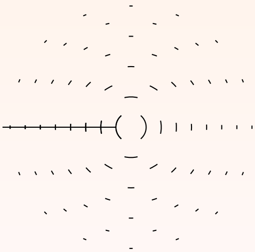

Equalities in non-simply connected cases

Theorem (Betsakos, C-Z – 2025)

Assume that the planar domain Ω has the following properties:

- Ω is unbounded.
- Let F be the union of all bounded components of $\mathbb{C}_\infty \setminus \Omega$. The set F is bounded.
- Consider the simply connected domain $\Omega' = \Omega \cup F$. For all sufficiently large $r > 0$, the set $\Omega' \cap \{z \in \mathbb{C} : |z| = r\}$ has exactly one component.

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Then,

$$\mathrm{h}(\Omega) = \mathrm{h}(\Omega') = \mathrm{b}(\Omega) = \mathrm{b}(\Omega') = \frac{\mathrm{b}_\alpha(\Omega)}{\alpha + 2} = \frac{\mathrm{b}_\alpha(\Omega')}{\alpha + 2}.$$

Holomorphic function spaces and the geometry of image domains

Francisco José Cruz Zamorano
Universidad de Sevilla (Spain)

Thank you!

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DEPARTAMENTO DE MATEMÁTICA APLICADA II

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